
UNIT 5 RANDOM VARIABLES AND EXPECTATION

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5.0 INTRODUCTION

In the previous units, we have studied the computation of probabilities of events in detail. In those units, we were interested in knowing the occurrence of outcomes. In the present unit, we will be interested in the numbers associated with such outcomes of the random experiments. Such an interest leads to studying the concept of random variable. In this unit, we will introduce the concept of random variable, discrete and continuous random variables their probability functions

There may also be situations where we have to study two-dimensional random variables in connection with a random experiment. For example, we may be interested in recording the number of boys and girls born in a hospital on a particular day. Here, 'the number of boys' and 'the number of girls' are random variables taking the values 0, 1, 2, ..., and both these random variables are also discrete. In this unit, we define two-dimensional discrete and continuous random variables. The joint, marginal and conditional probability mass functions of two-dimensional discrete random variables. The Unit also defines the distribution function and the marginal distribution function for bivariate continuous random variable

You have studied the methods of finding mean, variance and other measures in context of frequency distributions in Block 1 of this course. In the present unit, we discuss the expectations of random variables and their properties. Addition and multiplication laws of expectation have also been defined.

5.1 OBJECTIVES

A study of this unit would enable you to:

- define a random variable, discrete and continuous random variables;

- specify the probability mass function, i.e. probability distribution of discrete random variables;
- specify the probability density function, i.e. probability function of continuous random variables;
- define the distribution function and two two-dimensional discrete and continuous random variables;
- find the expected values of random variables;
- obtain various measures for probability distributions; and

5.2 RANDOM VARIABLE

The last unit discusses the process of computing probabilities for events (subsets of a sample space). In many experiments, we may be interested in a numerical characteristic associated with outcomes of a random experiment. Like the outcome, the value of such a numerical characteristic cannot be predicted in advance.

For example, suppose a die is tossed twice and we are interested in number of times an odd number appears. Let X be the number of appearances of odd number. If a die is thrown twice, an odd number may appear '0' times (i.e. we may have even number both the times) or once (i.e. we may have odd number in one throw and even number in the other throw) or twice (i.e. we may have odd number both the times). Here, X can take the values 0, 1, 2 and is a variable quantity behaving randomly and hence we may call it a 'random variable'. Also notice that its values are real and are defined on the sample space

$\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),$
 $(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),$
 $(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),$
 $(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),$
 $(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),$
 $(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

i.e.

$$X = \begin{cases} 0, & \text{if the outcome is } (2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), \\ & (6, 4), (6, 6) \\ 1, & \text{if the outcome is } (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 2), \\ & (3, 4), (3, 6), (4, 1), (4, 3), (4, 5), (5, 2), (5, 4), \\ & (5, 6), (6, 1), (6, 3), (6, 5) \\ 2, & \text{if the outcome is } (1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), \\ & (5, 3), (5, 5) \end{cases}$$

$$\text{So, } P[X = 0] = \frac{9}{36} = \frac{1}{4}, P[X = 1] = \frac{18}{36} = \frac{1}{2}, P[X = 2] = \frac{9}{36} = \frac{1}{4},$$

$$\text{and } P[X = 0] + P[X = 1] + P[X = 2] = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1.$$

Observe that a probability can be assigned to the event that X assumes a particular value. It can also be observed that the sum of the probabilities corresponding to different values of X is one.

So, a random variable can be defined as below:

Definition: A random variable is a real-valued function whose domain is a set of possible outcomes of a random experiment and range is a sub-set of the set of real numbers and has the following properties:

- i) Each particular value of the random variable can be assigned some probability
- ii) Uniting all the probabilities associated with all the different values of the random variable gives the value 1 (unity).

Remark 1: We shall denote random variables by capital letters like X, Y, Z , etc. and write **r.v.** for random variable.

5.3 DISCRETE RANDOM VARIABLE AND PROBABILITY MASS FUNCTION

Discrete Random Variable

A random variable is said to be discrete if it has either a finite or a countable number of values. Countable number of values means the values which can be arranged in a sequence, i.e. the values which have one-to-one correspondence with the set of natural numbers, i.e., on the basis of three or four successive known terms, we can catch a rule and hence can write the subsequent terms. For example, suppose X is a random variable taking the values say 2, 5, 8, 11, ... then we can write the fifth, sixth, ... values, because the values have one-to-one correspondence with the set of natural numbers and have the general term as $3n - 1$, i.e. on taking $n = 1, 2, 3, 4, 5, \dots$ we have 2, 5, 8, 11, 14, So, X in this example is a discrete random variable. The number of students present each day in a class during an academic session is an example of discrete random variable as the number cannot take a fractional value.

Probability Mass Function

Let X be a r.v. which takes the values $x_1, x_2, \dots$ and let $P[X = x_i] = p(x_i)$. This function $p(x_i)$, $i = 1, 2, \dots$ defined for the values $x_1, x_2, \dots$ assumed by X is called probability mass function of X satisfying $p(x_i) \geq 0$ and $\sum_i p(x_i) = 1$. The set

$\{(x_1, p(x_1)), (x_2, p(x_2)), \dots\}$ specifies the probability distribution of a discrete r.v. X . Probability distribution of r.v. X can also be exhibited in the following manner:

X	x_1	x_2	x_3	$\dots$
$p(x)$	$p(x_1)$	$p(x_2)$	$p(x_3)$	$\dots$

Now, let us take up some examples concerning probability mass function.

Example 1: State, giving reasons, which of the following are not probability distributions:

(i)

X	0	1
p(x)	$\frac{1}{2}$	$\frac{3}{4}$

(ii)

X	0	1	2
p(x)	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{3}{4}$

(iii)

X	0	1	2
p(x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(iv)

X	0	1	2	3
p(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

Solution:

(i) Here $p(x_i) \geq 0$, $i = 1, 2$; but

$$\sum_{i=1}^2 p(x_i) = p(x_1) + p(x_2) = p(0) + p(1) = \frac{1}{2} + \frac{3}{4} = \frac{5}{4} > 1.$$

So, the given distribution is not a probability distribution as $\sum_{i=1}^2 p(x_i)$ is greater than 1.

(ii) It is not probability distribution as $p(x_2) = p(1) = -\frac{1}{2}$ i.e. negative

(iii) Here, $p(x_i) \geq 0$, $i = 1, 2, 3$

$$\text{and } \sum_{i=1}^3 p(x_i) = p(x_1) + p(x_2) + p(x_3) = p(0) + p(1) + p(2) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1.$$

$\therefore$ The given distribution is probability distribution.

(iv) Here, $p(x_i) \geq 0$, $i = 1, 2, 3, 4$; but

$$\begin{aligned} \sum_{i=1}^4 p(x_i) &= p(x_1) + p(x_2) + p(x_3) + p(x_4) \\ &= p(0) + p(1) + p(2) + p(3) = \frac{1}{8} + \frac{3}{8} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} < 1. \end{aligned}$$

$\therefore$ The given distribution is not probability distribution.

Example 2: For the following probability distribution of a discrete r.v. X , find

- (i) the constant c ,
- (ii) $P[X \leq 3]$ and
- (iii) $P[1 < X < 4]$.

X	0	1	2	3	4	5
$p(x)$	0	c	c	$2c$	$3c$	c

Solution:

- i) As the given distribution is probability distribution,

$$\therefore \sum_i p(x_i) = 1$$

$$\Rightarrow 0 + c + c + 2c + 3c + c = 1 \Rightarrow 8c = 1 \Rightarrow c = \frac{1}{8}$$

- ii) $P[X \leq 3] = P[X = 3] + P[X = 2] + P[X = 1] + P[X = 0]$

$$= 2c + c + c + 0 = 4c = 4 \times \frac{1}{8} = \frac{1}{2}.$$

- iii) $P[1 < X < 4] = P[X = 2] + P[X = 3] = c + 2c = 3c = 3 \times \frac{1}{8} = \frac{3}{8}.$

Example 3: Find the probability distribution of the number of heads when three fair coins are tossed simultaneously.

Solution: Let X be the number of heads in the toss of three fair coins.

As the random variable, “the number of heads” in a toss of three coins may be 0 or 1 or 2 or 3 associated with the sample space

$\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\},$

$\therefore X$ can take the values 0, 1, 2, 3, with

$$P[X = 0] = P[TTT] = \frac{1}{8}$$

$$P[X = 1] = P[HTT, THT, TTH] = \frac{3}{8}$$

$$P[X = 2] = P[HHT, HTH, THH] = \frac{3}{8}$$

$$P[X = 3] = P[HHH] = \frac{1}{8}.$$

$\therefore$ Probability distribution of X , i.e. the number of heads when three coins are tossed simultaneously is

X	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

which is the required probability distribution.

Let us define and explain a continuous random variable and its probability function in the next section.

5.4 CONTINUOUS RANDOM VARIABLE AND PROBABILITY DENSITY FUNCTION

In Sec. 5.3 of this unit, we have defined the discrete random variable as a random variable having countable number of values, i.e. whose values can be arranged in a sequence. But, if a random variable is such that its values cannot be arranged in a sequence, it is called continuous random variable. Temperature of a city at various points of time during a day is an example of continuous random variable as the temperature takes uncountable values, i.e. it can take fractional values also. So, a random variable is said to be continuous if it can take all possible real (i.e. integer as well as fractional) values between two certain limits. For example, let us denote the variable, “Difference between the rainfall (in cm) of a city and that of another city on every rainy day in a rainy season”, by X , then X here is a continuous random variable as it can take any real value between two certain limits. It can be noticed that for a continuous random variable, the chance of occurrence of a particular value of the variable is very small, so instead of specifying the probability of taking a particular value by the variable, we specify the probability of its lying within an interval. For example, chance that an athlete will finish a 100-meter race in say exactly 10 seconds is very-very small, i.e. almost zero as it is very rare to finish the race in a fixed time. Here, the probability is specified for an interval, i.e. we may be interested in finding as to what is the probability of finishing the race by the athlete in an interval of say 10 to 12 seconds.

So, continuous random variable is represented by different representation known as **probability density function** unlike the discrete random variable which is represented by probability mass function.

Probability Density Function

Let $f(x)$ be a continuous function of x . Suppose the shaded region ABCD shown in the following figure represents the area bounded by $y = f(x)$, x -axis and the ordinates at the points x and $x + \delta x$, where δx is the length of the interval $(x, x + \delta x)$.

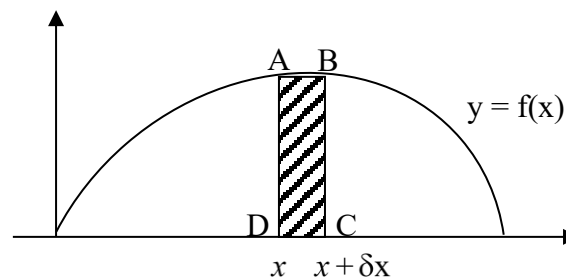


Fig. 5.1: Probability Density Function

Now, if δx is very-very small, then the curve AB will act as a line and hence the shaded region will be a rectangle whose area will be $AD \times DC$ i.e. $f(x) \delta x$

[$\because AD =$ the value of y at x i.e. $f(x)$, $DC =$ length δx of the interval $(x, x + \delta x)$]

Also, this area = probability that X lies in the interval $(x, x + \delta x)$

$$= P[x \leq X \leq x + \delta x]$$

Hence,

$$P[x \leq X \leq x + \delta x] = f(x) \delta x$$

$$\Rightarrow \frac{P[x \leq X \leq x + \delta x]}{\delta x} = f(x), \text{ where } \delta x \text{ is very-very small}$$

$$\Rightarrow \lim_{\delta x \rightarrow 0} \frac{P[x \leq X \leq x + \delta x]}{\delta x} = f(x).$$

$f(x)$, so defined, is called probability density function.

Probability density function has the same properties as that of probability mass function. So, $f(x) \geq 0$ and sum of the probabilities of all possible values that the random variable can take, has to be 1. But, here, as X is a continuous random variable, the summation is made possible through 'integration' and hence

$$\int_R f(x) dx = 1,$$

where integral has been taken over the entire range R of values of X .

Remark 2

- i) Summation and integration have the same meanings but in mathematics there is still difference between the two and that is that the former is used in case of discrete values, i.e. countable values and the latter is used in continuous case.
- ii) An essential property of a continuous random variable is that there is zero probability that it takes any specified numerical value, but the probability that it takes a value in specified intervals is non-zero and is calculable as a definite integral of the probability density function of the random variable and hence the probability that a continuous r.v. X will lie between two values a and b is given by

$$P[a < X < b] = \int_a^b f(x) dx.$$

Example 4: A continuous random variable X has the probability density function: $f(x) = Ax^3$, $0 \leq x \leq 1$. Determine

- i) A
- ii) $P[0.2 < X < 0.5]$

Solution:

(i) As $f(x)$ is probability density function,

$$\therefore \int_R f(x) dx = 1$$

$$\Rightarrow \int_0^1 f(x) dx = 1 \Rightarrow \int_0^1 Ax^3 dx = 1$$

$$\Rightarrow A \left[\frac{x^4}{4} \right]_0^1 = 1 \Rightarrow A \left(\frac{1}{4} - 0 \right) = 1 \Rightarrow A = 4$$

$$\begin{aligned} \text{(ii) } P[0.2 < X < 0.5] &= \int_{0.2}^{0.5} f(x) dx = \int_{0.2}^{0.5} Ax^3 dx = 4 \left[\frac{x^4}{4} \right]_{0.2}^{0.5} = [(0.5)^4 - (0.2)^4] \\ &= 0.0625 - 0.0016 = 0.0609 \end{aligned}$$

5.5 DISTRIBUTION FUNCTION

A function F defined for all values of a random variable X by $F(x) = P[X \leq x]$ is called the distribution function. It is also known as the cumulative distribution function (c.d.f.) of X since it is the cumulative probability of X up to and including the value x . As X can take any real value, therefore the domain of the distribution function is set of real numbers and as $F(x)$ is a probability value, therefore the range of the distribution function is $[0, 1]$.

Remark 3: Here, X denotes the random variable and x represents a particular value of random variable. $F(x)$ may also be written as $F_X(x)$, which means that it is a distribution function of random variable X .

Discrete Distribution Function

Distribution function of a discrete random variable is said to be discrete distribution function or cumulative distribution function (c.d.f.). Let X be a discrete random variable taking the values $x_1, x_2, x_3, \dots$ with respective probabilities $p_1, p_2, p_3, \dots$

$$\begin{aligned} \text{Then } F(x_i) &= P[X \leq x_i] = P[X = x_1] + P[X = x_2] + \dots + P[X = x_i] \\ &= p_1 + p_2 + \dots + p_i. \end{aligned}$$

The distribution function of X , in this case, is given as in the following table:

X	$F(x)$
x_1	p_1
x_2	$p_1 + p_2$
x_3	$p_1 + p_2 + p_3$
$\dots$	$\dots$
x_i	$p_1 + p_2 + p_3 + \dots + p_i$

The value of $F(x)$ corresponding to the last value of the random variable X is always 1, as it is the sum of all the probabilities. $F(x)$ remains 1 beyond this last value of X also, as it being a probability, it can never exceed one.

For example, let X be a random variable having the following probability distribution:

X	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Notice that $p(x)$ will be zero for other values of X . Then, Distribution function of X is given by

X	$F(x) = P[X \leq x]$
0	$\frac{1}{4}$
1	$\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$
2	$\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$

Here, for the last value, i.e. for $X = 2$, we have $F(x) = 1$.

Also, if we take a value beyond 2 say 4, then we get

$$\begin{aligned} F(4) &= P[X \leq 4] \\ &= P[X = 4] + P[X = 3] + P[X \leq 2] \\ &= 0 + 0 + 1 = 1. \end{aligned}$$

Example 5: A random variable X has the following probability function:

X	0	1	2	3	4	5	6	7
$p(x)$	0	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{1}{50}$	$\frac{17}{100}$

Determine the distribution function of X .

Solution: Here,

$$F(0) = P[X \leq 0] = P[X = 0] = 0,$$

$$F(1) = P[X \leq 1] = P[X = 0] + P[X = 1] = 0 + \frac{1}{10} = \frac{1}{10},$$

$$F(2) = P[X \leq 2] = P[X = 0] + P[X = 1] + P[X = 2] = 0 + \frac{1}{10} + \frac{1}{5} = \frac{3}{10},$$

and so on.

Continuous Distribution Function

Distribution function of a continuous random variable is called the continuous distribution function or cumulative distribution function (c.d.f.).

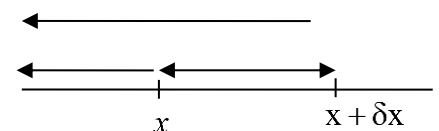
Let X be a continuous random variable having the probability density function $f(x)$, as defined in the last section of this unit, then the distribution function $F(x)$ is given by

$$F(x) = P[X \leq x] = \int_{-\infty}^x f(x) dx.$$

Also, in the last section, we have defined the p.d.f. $f(x)$ as

$$f(x) = \lim_{\delta x \rightarrow 0} \frac{P[x \leq X \leq x + \delta x]}{\delta x},$$

$$\therefore f(x) = \lim_{\delta x \rightarrow 0} \frac{P[X \leq x + \delta x] - P[X \leq x]}{\delta x}$$



$$\Rightarrow f(x) = \lim_{\delta x \rightarrow 0} \frac{F(x + \delta x) - F(x)}{\delta x},$$

$$\Rightarrow f(x) = \text{Derivative of } F(x) \text{ with respect to } x \quad \left[\begin{array}{l} \text{By definition of} \\ \text{the derivative} \end{array} \right]$$

$$\Rightarrow f(x) = F'(x)$$

$$\Rightarrow f(x) = \frac{d}{dx}(F(x))$$

$$\Rightarrow dF(x) = f(x) dx$$

Here, $dF(x)$ is known as the probability differential.

$$\text{So, } F(x) = \int_{-\infty}^x f_x(x) dx \text{ and } F'(x) = f(x).$$

Example 6: The diameter 'X' of a cable is assumed to be a continuous random variable with p.d.f. $f(x) = \begin{cases} 6x(1-x), & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$.

Obtain the c.d.f. of X.

Solution: For $0 \leq x \leq 1$, the c.d.f. of X is given by

$$\begin{aligned} F(x) &= P[X \leq x] = \int_0^x f(x) dx \\ &= \int_0^x 6x(1-x) dx \\ &= 6 \int_0^x (x - x^2) dx \\ &= 6 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^x = 3x^2 - 2x^3 \end{aligned}$$

$\therefore$ The c.d.f. of X is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ 3x^2 - 2x^3, & 0 \leq x \leq 1. \\ 1, & x > 1 \end{cases}$$

Remark 4: In the above example, $F(x)$ is taken as 0 for $x < 0$ since $p(x) = 0$ for $x < 0$; and $F(x)$ is taken as 1 for $x > 1$ since $F(1) = 1$ and therefore, for $x > 1$ also $F(x)$ will remain 1.

Check Your Progress 1

- 1) 2 bad articles are mixed with 5 good ones. Find the probability distribution of the number of bad articles, if 2 articles are drawn at random.

.....
.....

- 2) Given the probability distribution:

X	0	1	2	3
p(x)	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{10}$

Let $Y = X^2 + 2X$. Find the probability distribution of Y.

.....
.....

- 3) An urn contains 3 white and 4 red balls. 3 balls are drawn one by one with replacement. Find the probability distribution of the number of red balls drawn.

.....
.....

- 4) The life (in hours) X of a certain type of light bulb may be supposed to be a continuous random variable with p.d.f.:

$$f(x) = \begin{cases} \frac{A}{x^3}, & 1500 < x < 2500 \\ 0, & \text{elsewhere} \end{cases}$$

Determine the constant A and compute the probability that $1600 \leq X \leq 2000$.

.....
.....

- 5) A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7	8
p(x)	k	3k	5k	7k	9k	11k	13k	15k	17k

- (i) Determine the value of k.
(ii) Find the distribution function of X.

.....
.....

- 6) Let X be continuous random variable with p.d.f. given by.

$$f(x) = \begin{cases} \frac{x}{2}, & 0 \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ \frac{1}{2}(3-x), & 2 \leq x < 3 \\ 0, & \text{elsewhere.} \end{cases}$$

Determine F(x), the c.d.f. of X.

.....
.....

5.6 BIVARIATE RANDOM VARIABLES

In the previous sections, the concept of a single-dimensional random variable has been studied. Proceeding in analogy with the one-dimensional case, is the concept of two-dimensional random variables

5.6.1 Bivariate Discrete Random Variables

Definition: Let X and Y be two discrete random variables defined on the sample space S of a random experiment then the function (X, Y) defined on the same sample space is called a two-dimensional discrete random variable. In other words, (X, Y) is a two-dimensional random variable if the possible values of (X, Y) are finite or countably infinite. Here, each value of X and Y is represented as a point (x, y) in the xy -plane.

As an illustration, let us consider the following example:

Let three balls b_1, b_2, b_3 be placed randomly in three cells. The possible outcomes of placing the three balls in three cells are shown in Table 5.1.

Table 5.1: Possible Outcomes of Placing the Three Balls in Three Cells

Arrangement Number	Placement of the Balls in		
	Cell 1	Cell 2	Cell 3
1	b_1	b_2	b_3
2	b_1	b_3	b_2
3	b_2	b_1	b_3
4	b_2	b_3	b_1
5	b_3	b_1	b_2
6	b_3	b_2	b_1
7	b_1, b_2	b_3	-
8	b_1, b_2	-	b_3
9	-	b_1, b_2	b_3
10	b_1, b_3	b_2	-
11	b_1, b_3	-	b_2
12	-	b_1, b_3	b_2
13	b_2, b_3	b_1	-
14	b_2, b_3	-	b_1
15	-	b_2, b_3	b_1
16	b_1	b_2, b_3	-
17	b_1	-	b_2, b_3
18	-	b_1	b_2, b_3
19	b_2	b_3, b_1	-
20	b_2	-	b_3, b_1
21	-	b_2	b_3, b_1
22	b_3	b_1, b_2	-
23	b_3	-	b_1, b_2
24	-	b_3	b_1, b_2

25	b_1, b_2, b_3	-	-
26	-	b_1, b_2, b_3	-
27	-	-	b_1, b_2, b_3

Now, let X denote the number of balls in Cell 1 and Y be the number of cells occupied. Notice that X and Y are discrete random variables where X take on the values 0, 1, 2, 3 ($\because$ number of balls in Cell 1 may be 0 or 1 or 2 or 3) and Y take on the values 1, 2, 3 ($\because$ number of occupied cells may be 1 or 2 or 3). The possible values of two-dimensional random variable (X, Y) , therefore, are all ordered pairs of the values x and y of X and Y , respectively, i.e. are (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3).

Now, to each possible value (x_i, y_j) of (X, Y) , we can associate a number $p(x_i, y_j)$ representing $P(X = x_i, Y = y_j)$ as follows:

$p(0, 1) = P[X = 0, Y = 1] = P[\text{no ball in Cell 1 and 1 cell occupied}]$ $= P[\text{Arrangement numbers 26, 27}] = \frac{2}{27}$
$p(0, 2) = P[X = 0, Y = 2] = P[\text{no ball in Cell 1 and 2 cells occupied}]$ $= P[\text{Arrangement numbers 9, 12, 15, 18, 21, 24}]$ $= \frac{6}{27}$
$p(0, 3) = P[X = 0, Y = 3] = P[\text{no ball in Cell 1 and 3 cells occupied}]$ $= P[\text{Impossible event}] = 0$
$p(1, 1) = P[X = 1, Y = 1] = P[\text{one ball in Cell 1 and 1 cell occupied}]$ $= P[\text{Impossible event}] = 0$
$p(1, 2) = P[X = 1, Y = 2] = P[\text{one ball in Cell 1 and 2 cells occupied}]$ $= P[\text{Arrangement numbers 16, 17, 19, 20, 22, 23}]$ $= \frac{6}{27}$
$p(1, 3) = P[X = 1, Y = 3] = P[\text{one ball in Cell 1 and 3 cells occupied}]$ $= P[\text{Arrangement numbers 1 to 6}] = \frac{6}{27}$
$p(2, 1) = P[X = 2, Y = 1] = P[\text{two balls in Cell 1 and 1 cell occupied}]$ $= P[\text{Impossible event}] = 0$
$p(2, 2) = P[X = 2, Y = 2] = P[\text{two balls in Cell 1 and 2 cells occupied}]$ $= P[\text{Arrangement numbers 7, 8, 10, 11, 13, 14}]$ $= \frac{6}{27}$
$p(2, 3) = P[X = 2, Y = 3] = P[\text{two balls in Cell 1 and 3 cells occupied}]$ $= P[\text{Impossible event}] = 0$
$p(3, 1) = P[X = 3, Y = 1] = P[\text{three balls in Cell 1 and 1 cell occupied}]$ $= P[\text{Arrangement number 25}] = \frac{1}{27}$

$p(3, 2) = P[X = 3, Y = 2] = P[\text{three balls in Cell 1 and 2 cells occupied}]$ $= P[\text{Impossible event}] = 0$
$p(3, 3) = P[X = 3, Y = 3] = P[\text{three balls in Cell 1 and 3 cells occupied}]$ $= P[\text{Impossible event}] = 0$

The values of (X, Y) together with the number associated as above constitute what is known as joint probability distribution of (X, Y) which can be written in the tabular form also as shown below:

Table 5.2: The Joint Probability Distribution for Table 5.1

Y \ X	1	2	3	Total
0	2 / 27	6 / 27	0	8 / 27
1	0	6 / 27	6 / 27	12 / 27
2	0	6 / 27	0	6 / 27
3	1 / 27	0	0	1 / 27
Total	3 / 27	18 / 27	6 / 27	1

We are now in a position to define joint, marginal and conditional probability mass functions.

Joint Probability Mass Function

Let (X, Y) be a two-dimensional discrete random variable. With each possible outcome (x_i, y_j) , we associate a number $p(x_i, y_j)$ representing $P[X = x_i, Y = y_j]$ or $P[X = x_i \cap Y = y_j]$ and satisfying following conditions:

$$(i) \quad p(x_i, y_j) \geq 0$$

$$(ii) \quad \sum_i \sum_j p(x_i, y_j) = 1$$

The function p defined for all (x_i, y_j) is in analogy with one-dimensional case and called the **joint probability mass function of X and Y**. It is usually represented in the form of the table as shown in the Table 5.2.

Marginal Probability Function

Let (X, Y) be a discrete two-dimensional random variable which take up finite or countably infinite values (x_i, y_j) . For each such two-dimensional random variable (X, Y), we may be interested in the probability distribution of X or the probability distribution of Y, individually.

Let us again consider the example of the random placement of three balls in there cells wherein X and Y are the discrete random variables representing “the number of balls in Cell 1” and “the number of occupied cells”, respectively. Let us consider Table 5.2 showing the joint distribution of (X, Y). From this table, let us take up the row totals and column totals. The row totals in the table represent the probability distribution of X and the column totals represent the probability distribution of Y, individually. That is,

$$P[X = 0] = \frac{2}{27} + \frac{6}{27} + 0 = \frac{8}{27}$$

$$P[X = 1] = 0 + \frac{6}{27} + \frac{6}{27} = \frac{12}{27}$$

$$P[X = 2] = 0 + \frac{6}{27} + 0 = \frac{6}{27}$$

$$P[X = 3] = \frac{1}{27} + 0 + 0 = \frac{1}{27}$$

and

$$P[Y = 1] = \frac{2}{27} + 0 + 0 + \frac{1}{27} = \frac{3}{27}$$

$$P[Y = 2] = \frac{6}{27} + \frac{6}{27} + \frac{6}{27} + 0 = \frac{18}{27}$$

$$P[Y = 3] = 0 + \frac{6}{27} + 0 + 0 = \frac{6}{27}$$

These distributions of X and Y , individually, are called the **marginal probability distributions** of X and Y , respectively.

So, if (X, Y) is a discrete two-dimensional random variable which take up the values (x_i, y_j) , then the probability distribution of X is determined as follows:

$$\begin{aligned} p(x_i) &= P[X = x_i] \\ &= P[(X = x_i \cap Y = y_1) \text{ or } (X = x_i \cap Y = y_2) \text{ or } \dots] \\ &= P[X = x_i \cap Y = y_1] + P[X = x_i \cap Y = y_2] + P[X = x_i \cap Y = y_3] + \dots \\ &= \sum_j P[X = x_i \cap Y = y_j] \\ &= \sum_j p(x_i, y_j) \left[\begin{array}{l} \because p(x_i, y_j), \text{ the joint probability mass} \\ \text{function, is } P[X = x_i \cap Y = y_j] \end{array} \right] \end{aligned}$$

which is known as the marginal probability mass function of X . Similarly, the probability distribution of Y is

$$\begin{aligned} p(y_j) &= P[Y = y_j] \\ &= P[X = x_1 \cap Y = y_j] + P[X = x_2 \cap Y = y_j] + \dots \\ &= \sum_i P[X = x_i \cap Y = y_j] \\ &= \sum_i p(x_i, y_j) \end{aligned}$$

and is known as the marginal probability mass function of Y .

5.6.2 Bivariate Continuous Random Variables

Definition: If X and Y are continuous random variables defined on the sample space S of a random experiment, then (X, Y) defined on the same sample space S is called bivariate continuous random variable if (X, Y) assigns a point in xy -plane defined on the sample space S . Notice that it (unlike discrete random

variable) assumes values in some non-countable set. Some examples of bivariate continuous random variables are:

1. A gun is aimed at a certain point (say origin of the coordinate system). Because of the random factors, suppose the actual hit point is any point (X, Y) in a circle of radius unity about the origin.

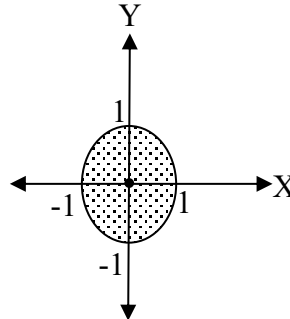


Fig. 5.2: Actual Hit Point When a Gun is Aimed at a Certain Point

Then (X, Y) assumes all the values in the circle $\{(x, y) : x^2 + y^2 \leq 1\}$ i.e. (X, Y) assumes all values corresponding to each and every point in the circular region as shown in Fig. 5.2. Here, (X, Y) is bivariate continuous random variable.

2. Assuming all values in the rectangle, represented as (X, Y) $\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$ is a bivariate continuous random variable. Here, (X, Y) assumes all values corresponding to each and every point in the rectangular region as shown in Fig. 5.3.

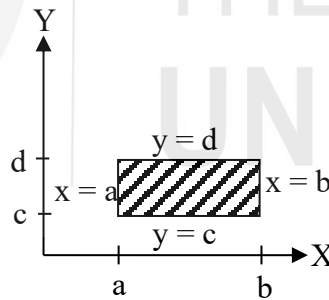


Fig. 5.3: (X, Y) Assuming All Values in the Rectangle $\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$

3. In a statistical survey, let X denotes the daily number of hours a child watches television and Y denotes the number of hours he/she spends on studies. Here, (X, Y) is a two-dimensional continuous random variable.

Joint and marginal distribution and density functions for continuous variables are defined using integration formulas. A detailed discussion on these topics is beyond the scope of this Unit. You may refer to further reading for more details on these topics.

Check Your Progress 2

- 1) The joint probability distribution of a pair of random variables is given by the following table:

X \ Y	1	2	3
1	1/12	0	1/18
2	1/6	1/9	1/4
3	0	1/5	2/15

(i) Evaluate marginal distribution of X.

(ii) Obtain $P[X + Y < 5]$.

.....
.....

2) For the following joint probability distribution of (X, Y),

X \ Y	1	2	3
1	1/20	1/10	1/10
2	1/20	1/10	1/10
3	1/10	1/10	1/20
4	1/10	1/10	1/20

(i) find the probability that $Y = 2$ given that $X = 4$,

(ii) find the probability that $Y = 2$

.....
.....

3) Let the joint density function of a two-dimensional random variable (X, Y) be:

$$f(x, y) = \begin{cases} x + y & \text{for } 0 \leq x < 1 \text{ and } 0 \leq y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the conditional density function of Y given X.

.....
.....

5.7 EXPECTATION OF A RANDOM VARIABLE

Let us now define a similar measure for the probability distribution of a random variable X, which assumes the values say $x_1, x_2, \dots, x_n$ with their associated probabilities $p_1, p_2, \dots, p_n$. This measure is known as the expected value of X and is given as

$$x_1(p_1) + x_2(p_2) + \dots + x_n(p_n) = \sum_{i=1}^n x_i p_i$$

The expected value of X is written as $E(X)$.

The above aspect can be viewed in the following way also:

Mean of a frequency distribution of X is $\frac{\sum_{i=1}^n x_i f_i}{\sum_{i=1}^n f_i}$, similarly mean of a

probability distribution of r.v. X is $\frac{\sum_{i=1}^n x_i p_i}{\sum_{i=1}^n p_i}$.

Now, as we know that $\sum_{i=1}^n p_i = 1$ for a probability distribution, therefore the

mean of the probability distribution becomes $\sum_{i=1}^n x_i p_i$.

$\therefore$ Expected value of a random variable X is $E(X) = \sum_{i=1}^n x_i p_i$.

The above formula for finding the expected value of a random variable X is used only if X is a discrete random variable which takes the values $x_1, x_2, \dots, x_n$ with probability mass function $p(x_i) = P[X = x_i]$, $i = 1, 2, \dots, n$.

But, if X is a continuous random variable having the probability density function $f(x)$, then in place of summation we will use integration and in this case, the expected value of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx,$$

The expectation, as defined above, agrees with the logical/theoretical argument also as is illustrated in the following example.

Suppose, a fair coin is tossed twice, then answer to the question, “How many heads do we expect theoretically/logically in two tosses?” is obviously 1 as the coin is unbiased and hence we will undoubtedly expect one head in two tosses. Expectation actually means “what we get on an average”? Now, let us obtain the expected value of the above question using the formula.

Let X be the number of heads in two tosses of the coin and we are to obtain $E(X)$, i.e. expected number of heads. As X is the number of heads in two tosses of the coin, therefore X can take the values 0, 1, 2 and its probability distribution is given as

$$\begin{array}{ccc} X: & 0 & 1 & 2 \\ p(x): & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{array}$$

$$\therefore E(X) = \sum_{i=1}^3 x_i p_i$$

$$= x_1 p_1 + x_2 p_2 + x_3 p_3$$

$$= (0)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{4}\right) = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

So, we get the same answer, i.e. 1 using the formula also.

So, expectation of a random variable is nothing but the average (mean) taken over all the possible values of the random variable or it is the value which we get on an average when a random experiment is performed repeatedly.

Remark 5: Sometimes summations and integrals as considered in the above definitions may not be convergent and hence expectations in such cases do not exist. But we will deal only those summations (series) and integrals which are convergent as the topic regarding checking the convergence of series or integrals is out of the scope of this course. You need not to bother as to whether the series or integral is convergent or not, i.e. as to whether the expectation exists or not as we are dealing with only those expectations which exist.

Example 7: If it rains, a raincoat dealer can earn Rs 500 per day. If it is a dry day, he can lose Rs 100 per day. What is his expectation, if the probability of rain is 0.4?

Solution: Let X be the amount earned on a day by the dealer. Therefore, X can take the values Rs 500, – Rs 100 ($\because$ loss of Rs 100 is equivalent to negative of the earning of Rs 100).

$\therefore$ Probability distribution of X is given as

	Rainy Day	Dry day
X (in Rs.):	500	–100
$p(x)$:	0.4	0.6

Hence, the expectation of the amount earned by him is

$$\begin{aligned} E(X) &= \sum_{i=1}^2 x_i p_i = x_1 p_1 + x_2 p_2 \\ &= (500)(0.4) + (-100)(0.6) = 200 - 60 = 140 \end{aligned}$$

Thus, his expectation is Rs 140, i.e. on an average he earns Rs 140 per day.

Example 8: A player tosses two unbiased coins. He wins Rs 5 if 2 heads appear, Rs 2 if one head appears and Rs 1 if no head appears. Find the expected value of the amount won by him.

Solution: In tossing two unbiased coins, the sample space, is

$$S = \{HH, HT, TH, TT\}.$$

$$\therefore P[2 \text{ heads}] = \frac{1}{4}, \quad P(\text{one head}) = \frac{2}{4}, \quad P(\text{no head}) = \frac{1}{4}.$$

Let X be the amount in rupees won by him

$\therefore X$ can take the values 5, 2 and 1 with

$$P[X = 5] = P(2 \text{ heads}) = \frac{1}{4},$$

$$P[X = 2] = P[1 \text{ Head}] = \frac{2}{4}, \text{ and}$$

$$P[X = 1] = P[\text{no Head}] = \frac{1}{4}.$$

∴ Probability distribution of X is

X:	5	2	1
p(x)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Expected value of X is given as

$$\begin{aligned} E(X) &= \sum_{i=1}^3 x_i p_i = x_1 p_1 + x_2 p_2 + x_3 p_3 \\ &= 5\left(\frac{1}{4}\right) + 2\left(\frac{2}{4}\right) + 1\left(\frac{1}{4}\right) = \frac{5}{4} + \frac{4}{4} + \frac{1}{4} = \frac{10}{4} = 2.5. \end{aligned}$$

Thus, the expected value of amount won by him is Rs 2.5.

Example 9: Find the expectation of the number on an unbiased die when thrown.

Solution: Let X be a random variable representing the number on a die when thrown.

∴ X can take the values 1, 2, 3, 4, 5, 6 with

$$P[X = 1] = P[X = 2] = P[X = 3] = P[X = 4] = P[X = 5] = P[X = 6] = \frac{1}{6}.$$

Thus, the probability distribution of X is given by

X:	1	2	3	4	5	6
p(x):	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Hence, the expectation of number on the die when thrown is

$$E(X) = \sum_{i=1}^6 x_i p_i = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = \frac{21}{6} = \frac{7}{2}$$

Example 10: Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the expected value for the number of aces.

Solution: Let A_1, A_2 be the events of getting ace in first and second draws, respectively. Let X be the number of aces drawn. Thus, X can take the values 0, 1, 2 with

$$\begin{aligned} P[X = 0] &= P[\text{no ace}] = P[\bar{A}_1 \cap \bar{A}_2] \\ &= P[\bar{A}_1] P[\bar{A}_2] \quad \left[\begin{array}{l} \because \text{cards are drawn with replacement} \\ \text{and hence the events are independent} \end{array} \right] \\ &= \frac{48}{52} \times \frac{48}{52} = \frac{12}{13} \times \frac{12}{13} = \frac{144}{169}, \end{aligned}$$

$$\begin{aligned} P[X = 1] &= [\text{one Ace and one other card}] \\ &= P[(A_1 \cap \bar{A}_2) \cup (\bar{A}_1 \cap A_2)] \end{aligned}$$

$$= P[A_1 \cap \bar{A}_2] + P[\bar{A}_1 \cap A_2] \quad \left[\begin{array}{l} \text{By Addition theorem of probability} \\ \text{for mutually exclusive events} \end{array} \right]$$

$$= P[A_1]P[\bar{A}_2] + P[\bar{A}_1]P[A_2] \quad \left[\begin{array}{l} \text{By multiplication theorem of} \\ \text{probability for independent events} \end{array} \right]$$

$$= \frac{4}{52} \times \frac{48}{52} + \frac{48}{52} \times \frac{4}{52} = \frac{1}{13} \times \frac{12}{13} + \frac{12}{13} \times \frac{1}{13} = \frac{24}{169}, \text{ and}$$

$$P[X = 2] = P[\text{both aces}] = P[A_1 \cap A_2]$$

$$= P[A_1]P[A_2] = \frac{4}{52} \times \frac{4}{52} = \frac{1}{169}.$$

Hence, the probability distribution of random variable X is

$$\begin{array}{ccc} X: & 0 & 1 & 2 \\ p(x): & \frac{144}{169} & \frac{24}{169} & \frac{1}{169} \end{array}$$

∴ The expected value of X is given by

$$E(X) = \sum_{i=1}^3 x_i p_i = 0 \times \frac{144}{169} + 1 \times \frac{24}{169} + 2 \times \frac{1}{169} = \frac{26}{169} = \frac{2}{13}$$

Example 11: For a continuous distribution whose probability density function is given by:

$$f(x) = \frac{3x}{4}(2-x), \quad 0 \leq x \leq 2, \text{ find the expected value of X.}$$

Solution: Expected value of a continuous random variable X is given by

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx = \int_0^2 x \frac{3x}{4}(2-x)dx = \frac{3}{4} \int_0^2 x^2(2-x)dx \\ &= \frac{3}{4} \int_0^2 (2x^2 - x^3)dx = \frac{3}{4} \left[2 \frac{x^3}{3} - \frac{x^4}{4} \right]_0^2 = \frac{3}{4} \left[2 \frac{(2)^3}{3} - \frac{(2)^4}{4} - 0 \right] \\ &= \frac{3}{4} \left[\frac{16}{3} - \frac{16}{4} \right] = \frac{3}{4} \times \frac{16}{12} = 1 \end{aligned}$$

Let us now discuss some properties of expectation in the next section.

Properties of Expectation of One-Dimensional Random Variable

Properties of mathematical expectation of a random variable X are given below without proof. You can refer to references for the proof:

1. $E(k) = k$, where k is a constant
2. $E(kX) = kE(X)$, k being a constant.
3. $E(aX + b) = aE(X) + b$, where a and b are constants

Example 12: Given the following probability distribution:

X	-2	-1	0	1	2
p(x)	0.15	0.30	0	0.30	0.25

- Find
- i) $E(X)$
 - ii) $E(2X + 3)$
 - iii) $E(X^2)$
 - iv) $E(4X - 5)$

Solution

$$\begin{aligned} \text{i) } E(X) &= \sum_{i=1}^5 x_i p_i = x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 + x_5 p_5 \\ &= (-2)(0.15) + (-1)(0.30) + (0)(0) + (1)(0.30) + (2)(0.25) \\ &= -0.3 - 0.3 + 0 + 0.3 + 0.5 = 0.2 \end{aligned}$$

$$\begin{aligned} \text{ii) } E(2X + 3) &= 2E(X) + 3 && [\text{Using property 3 of Expectation Sec 5.7}] \\ &= 2(0.2) + 3 && [\text{Using solution (i) of the question}] \\ &= 0.4 + 3 = 3.4 \end{aligned}$$

$$\begin{aligned} \text{iii) } E(X^2) &= \sum_{i=1}^5 x_i^2 p_i && [\text{By def.}] \\ &= x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 + x_4^2 p_4 + x_5^2 p_5 \\ &= (-2)^2 (0.15) + (-1)^2 (0.30) + (0)^2 (0) + (1)^2 (0.30) + (2)^2 (0.25) \\ &= (4)(0.15) + (1)(0.30) + (0) + (1)(0.30) + (4)(0.25) \\ &= 0.6 + 0.3 + 0 + 0.3 + 1 = 2.2 \end{aligned}$$

$$\begin{aligned} \text{iv) } E(4X - 5) &= E[4X + (-5)] \\ &= 4E(X) + (-5) && [\text{Using property 3 of Expectation Sec 5.7}] \\ &= 4(0.2) - 5 \\ &= 0.8 - 5 = -4.2 \end{aligned}$$

Check Your Progress 3

- 1) You toss a fair coin. If the outcome is head, you win Rs 100; if the outcome is tail, you win nothing. What is the expected amount won by you?

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.....

- 2) A fair coin is tossed until a tail appears. What is the expectation of number of tosses?

.....
.....

- 3) The distribution of a continuous random variable X is defined by

$$f(x) = \begin{cases} x^3 & , 0 < x \leq 1 \\ (2-x)^3 & , 1 < x \leq 2 \\ 0 & , \text{ elsewhere} \end{cases}$$

Obtain the expected value of X.

.....
.....

- 4) If X is a random variable with mean 'μ' and standard deviation 'σ', then what is the expectation of $Z = \frac{X - \mu}{\sigma}$?

[Note: Here Z so defined is called standard random variate.]

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5.8 MOMENTS AND OTHER MEASURES IN TERMS OF EXPECTATIONS

In this section, we present basic definitions of the moments and other measures for a random variable in terms of expectations in the following section

5.8.1 Moments

The moments for probability distributions for the r^{th} order moment about any point 'A' of a random variable X having probability mass function $P[X = x_i] = p(x_i) = p_i$ is defined as

$$\begin{aligned}\mu_r' &= \frac{\sum_{i=1}^n p_i (x_i - A)^r}{\sum_{i=1}^n p_i} \\ &= \sum_{i=1}^n p_i (x_i - A)^r \quad \left[\because \sum_{i=1}^n p_i = 1 \right]\end{aligned}$$

The above formula is valid if X is a discrete random variable. But, if X is a continuous random variable having probability density function $f(x)$, then

r^{th} order moment about A is defined as $\mu_r' = \int_{-\infty}^{\infty} (x - A)^r f(x) dx$.

So, r^{th} order moment about any point 'A' of a random variable X is defined as

$$\begin{aligned}\mu_r' &= \begin{cases} \sum_i p_i (x_i - A)^r, & \text{if X is a discrete r.v.} \\ \int_{-\infty}^{\infty} (x - A)^r f(x) dx, & \text{if X is a continuous r.v.} \end{cases} \\ &= E(X - A)^r\end{aligned}$$

Similarly, r^{th} order moment about mean (μ) i.e. r^{th} order central moment is defined as

$$\mu_r = \begin{cases} \sum_i p_i (x_i - \mu)^r, & \text{if } X \text{ is a discrete r.v.} \\ \int_{-\infty}^{\infty} (x - \mu)^r f(x) dx, & \text{if } X \text{ is a continuous r.v.} \end{cases}$$

$$= E(X - \mu)^r = E[X - E(X)]^r$$

5.8.2 Variance and Covariance

Variance of a random variable X is second order central moment and is defined as

$$\mu_2 = V(X) = E[X - \mu]^2 = E[X - E(X)]^2$$

Also, we know that

$$V(X) = \mu_2' - (\mu_1')^2$$

where μ_1', μ_2' be the moments about origin.

$$\therefore \text{We have } V(X) = E(X^2) - [E(X)]^2$$

$$\left[\because \mu_1' = E[X - 0]^1 = E(X), \text{ and } \mu_2' = E[X - 0]^2 = E(X^2) \right]$$

Covariance

For a bivariate frequency distribution, you have already studied in Unit 6 of MST-002 that covariance between two variables X and Y is defined as

$$\text{Cov}(X, Y) = \frac{\sum_i f_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i f_i}$$

$\therefore$ For a bivariate probability distribution, $\text{Cov}(X, Y)$ is defined as

$$\text{Cov}(X, Y) = \begin{cases} \sum_i p_{ij} (x_i - \bar{x})(y_j - \bar{y}), & \text{if } (X, Y) \text{ is two-dimensional discrete r.v.} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{x})(y - \bar{y}) f(x, y) dy dx, & \text{if } (X, Y) \text{ is two dimensional continuous r.v.} \end{cases}$$

$$\text{where } p_{ij} = P[X = x_i, Y = y_j]$$

$$= E(X - \bar{X})(Y - \bar{Y}) \quad [\text{By definition of expectation}]$$

$$= E[(X - E(X))(Y - E(Y))] \quad \left[\begin{array}{l} \because E(X) = \text{Mean of } X \text{ i.e. } \bar{X}, \\ E(Y) = \text{Mean of } Y \text{ i.e. } \bar{Y} \end{array} \right]$$

On simplifying,

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y).$$

Now, if X and Y are independent random variables then, by multiplication theorem,

$$E(XY) = E(X)E(Y) \text{ and hence in this case } \text{Cov}(X, Y) = 0.$$

Remark 6:

i) If X and Y are independent random variables, then

$$V(X + Y) = V(X) + V(Y).$$

ii) If X and Y are independent random variables, then

$$V(X - Y) = V(X) + V(Y).$$

iii) If X and Y are independent random variables, then

$$V(aX + bY) = a^2 V(X) + b^2 V(Y).$$

Mean Deviation about Mean

Mean deviation about mean in context of probability distribution is

$$\frac{\sum_{i=1}^n p_i |x_i - \text{mean}|}{\sum_{i=1}^n p_i} = \sum_{i=1}^n p_i |x_i - \text{mean}|$$

$\therefore$ by definition of expectation, we have

$$\text{M.D. about mean} = E|X - \text{Mean}|$$

$$= E|X - E(X)|$$

$$= \begin{cases} \sum p_i |x_i - \text{Mean}| & \text{for discrete r.v} \\ \int_{-\infty}^{\infty} |x - \text{Mean}| f(x) dx & \text{for continuous r.v} \end{cases}$$

Example 13: Considering the probability distribution given in Example 12, obtain

i) $V(X)$

ii) $V(2X + 3)$.

Solution:

$$(i) \quad V(X) = E(X^2) - [E(X)]^2$$

$$= 2.2 - (0.2)^2 \quad \left[\begin{array}{l} \text{The values have already been obtained} \\ \text{in the solution of Example 6} \end{array} \right]$$

$$= 2.2 - 0.04 = 2.16$$

$$(ii) \quad V(2X + 3) = (2)^2 V(X) \quad [\text{Using the result of Theorem 8.1}]$$

$$= 4V(X) = 4(2.16) = 8.64$$

Example 14: If X and Y are independent random variables with variances 2 and 3, respectively, find the variance of $3X + 4Y$.

$$\text{Solution: } V(3X + 4Y) = (3)^2 V(X) + (4)^2 V(Y)$$

$$= 9(2) + 16(3) = 66$$

5.8.3 Addition and Multiplication Theorems of Expectation

Two important properties, i.e., the addition and multiplication laws of expectation, are stated next (without withproof).

Addition Theorem of Expectation

Theorem 8.1: If X and Y are random variables, then $E(X + Y) = E(X) + E(Y)$

Remark 7: The result can be similarly extended for more than two random variables.

Multiplication Theorem of Expectation

Theorem 8.2: If X and Y are independent random variables, then

$$E(XY) = E(X) E(Y)$$

Remark 8: The result can be similarly extended for more than two random variables.

Example 15: Two unbiased dice are thrown. Find the expected value of the sum of number of points on them.

Solution: Let X be the number obtained on the first die and Y be the number obtained on the second die, then

$$E(X) = \frac{7}{2} \text{ and } E(Y) = \frac{7}{2}$$

$\therefore$ The required expected value $= E(X + Y)$

$$= E(X) + E(Y) \left[\begin{array}{l} \text{Using addition theorem} \\ \text{of expectation} \end{array} \right]$$

$$= \frac{7}{2} + \frac{7}{2} = 7$$

Remark 9: This example can also be done considering one random variable only as follows:

Let X be the random variable denoting “the sum of numbers of points on the dice”, then the probability distribution in this case is

$X:$	2	3	4	5	6	7	8	9	10	11	12
$p(x):$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\text{and hence } E(X) = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + \dots + 12 \times \frac{1}{36} = 7$$

Example 16: Two cards are drawn one by one with replacement from 8 cards numbered from 1 to 8. Find the expectation of the product of the numbers on the drawn cards.

Solution: Let X be the number on the first card and Y be the number on the second card. Then probability distribution of X is

X	1	2	3	4	5	6	7	8
$p(x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

and the probability distribution of Y is

Y	1	2	3	4	5	6	7	8
p(y)	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

$$\begin{aligned}\therefore E(X) &= E(Y) = 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + \dots + 8 \times \frac{1}{8} \\ &= \frac{1}{8}(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8) = \frac{1}{8}(36) = \frac{9}{2}\end{aligned}$$

Thus, the required expected value is

$$E(XY) = E(X)E(Y) \quad [\text{Using multiplication theorem of expectation}]$$

$$= \frac{9}{2} \times \frac{9}{2} = \frac{81}{4}.$$

Check Your Progress 4

- 1) If X is a random variable with mean μ and standard deviation σ , then find the variance of standard random variable $Z = \frac{X - \mu}{\sigma}$.

.....
.....

- 2) Suppose that X is a random variable for which $E(X) = 10$ and $V(X) = 25$. Find the positive values of a and b such that $Y = aX - b$ has expectation.

.....
.....

5.9 SUMMARY

Following main points have been covered in this unit of the course:

- 1) A **random variable** is a function whose domain is a set of possible outcomes and range is a sub-set of the set of reals and has the following properties:
 - i) Each particular value of the random variable can be assigned some probability.
 - ii) Sum of all the probabilities associated with all the different values of the random variable is unity.
- 2) A random variable is said to be **discrete random variable** if it has either a finite number of values or a countable number of values, i.e. the values can be arranged in a sequence.
- 3) If a random variable is such that its values cannot be arranged in a sequence, it is called **continuous random variable**. So, a random variable is said to be continuous if it can take all the possible real (i.e. integer as well as fractional) values between two certain limits.

- 4) It discussed the concept of **probability mass function** and defined the **probability distribution** of discrete random variable and **probability density function** of a continuous random variable
- 5) The unit also defined the **distribution function** also known as the **cumulative distribution function (c.d.f.)** of a random variable. It explains **discrete distribution function** and **continuous distribution function**.
- 6) The unit briefly presents concepts of **two-dimensional discrete random variable** and **bivariate continuous random variable**.
- 7) The unit defined the Expected value of a random variable and its properties.
- 8) It defines Moments, variance and covariance in terms of expectation.
- 9) The unit has presented many related example and questions. If you want to see what our solutions to the check your progress questions, we have given them in the following section.

5.10 SOLUTIONS/ANSWERS

Check Your Progress 1

- 1) Let X be the number of bad articles drawn.

$\therefore$ X can take the values 0, 1, 2 with

$$P[X = 0] = P[\text{No bad article}]$$

= P[Drawing 2 articles from 5 good articles and zero article
from 2 bad articles]

$$= \frac{{}^5C_2 \times {}^2C_0}{{}^7C_2} = \frac{5 \times 4 \times 1}{7 \times 6} = \frac{10}{21},$$

$$P[X = 1] = P[\text{One bad article and 1 good article}]$$

$$= \frac{{}^2C_1 \times {}^5C_1}{{}^7C_2} = \frac{2 \times 5 \times 2}{7 \times 6} = \frac{10}{21},$$

And

$$P[X = 2] = P[\text{Two bad articles and no good article}]$$

$$= \frac{{}^2C_2 \times {}^5C_0}{{}^7C_2} = \frac{1 \times 1 \times 2}{7 \times 6} = \frac{1}{21}$$

$\therefore$ Probability distribution of number of bad articles is:

X	0	1	2
p(x)	$\frac{10}{21}$	$\frac{10}{21}$	$\frac{1}{21}$

2) As $Y = X^2 + 2X$,

$\therefore$ For $X = 0$, $Y = 0 + 0 = 0$;

For $X = 1$, $Y = 1^2 + 2(1) = 3$;

For $X = 2$, $Y = 2^2 + 2(2) = 8$; and

For $X = 3$, $Y = 3^2 + 2(3) = 15$.

Thus, the values of Y are 0, 3, 8, 15 corresponding to the values 0, 1, 2, 3 of X and hence

$$P[Y = 0] = P[X = 0] = \frac{1}{10}, P[Y = 3] = P[X = 1] = \frac{3}{10},$$

$$P[Y = 8] = P[X = 2] = \frac{1}{2} \text{ and } P[Y = 15] = P[X = 3] = \frac{1}{10}.$$

$\therefore$ The probability distribution of Y is

Y	0	3	8	15
p(y)	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{10}$

3) Let X be the number of red balls drawn.

$\therefore$ X can take the values 0, 1, 2, 3.

Let W_i be the event that i^{th} draw gives white ball and R_i be the event that i^{th} draw gives red ball.

$$\begin{aligned} \therefore P[X = 0] &= P[\text{No Red ball}] = P[W_1 \cap W_2 \cap W_3] \\ &= P(W_1) \cdot P(W_2) \cdot P(W_3) \end{aligned}$$

[$\because$ balls are drawn with replacement and hence the draws are independent]

$$= \frac{3}{7} \times \frac{3}{7} \times \frac{3}{7} = \frac{27}{343}$$

$$P[X = 1] = P[\text{One red and two white}]$$

$$= P[(R_1 \cap W_2 \cap W_3) \text{ or } (W_1 \cap R_2 \cap W_3) \text{ or } (W_1 \cap W_2 \cap R_3)]$$

$$= P[R_1 \cap W_2 \cap W_3] + P[W_1 \cap R_2 \cap W_3] + P[W_1 \cap W_2 \cap R_3]$$

$$= P[R_1]P[W_2]P[W_3] + P[W_1]P[R_2]P[W_3] + P[W_1]P[W_2]P[R_3]$$

$$= \frac{4}{7} \times \frac{3}{7} \times \frac{3}{7} + \frac{3}{7} \times \frac{4}{7} \times \frac{3}{7} + \frac{3}{7} \times \frac{3}{7} \times \frac{4}{7} = 3 \times \frac{4}{7} \times \frac{3}{7} \times \frac{3}{7} = \frac{108}{343},$$

$$P[X = 2] = P[\text{Two red and one white}]$$

$$= P[(R_1 \cap R_2 \cap W_3) \text{ or } (R_1 \cap W_2 \cap R_3) \text{ or } (W_1 \cap R_2 \cap R_3)]$$

$$= P[R_1]P[R_2]P[W_3] + P[R_1]P[W_2]P[R_3] + P[W_1]P[R_2]P[R_3]$$

$$= \frac{4}{7} \times \frac{4}{7} \times \frac{3}{7} + \frac{4}{7} \times \frac{3}{7} \times \frac{4}{7} + \frac{3}{7} \times \frac{4}{7} \times \frac{4}{7} = 3 \times \frac{4}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{144}{343}.$$

$$P[X = 3] = P[\text{Three red balls}]$$

$$= P[R_1 \cap R_2 \cap R_3] = P(R_1) P(R_2) P(R_3) = \frac{4}{7} \times \frac{4}{7} \times \frac{4}{7} = \frac{64}{343}.$$

$\therefore$ Probability distribution of the number of red balls is

X	0	1	2	3
p(x)	$\frac{27}{343}$	$\frac{108}{343}$	$\frac{144}{343}$	$\frac{64}{343}$

4) As $f(x)$ is p.d.f.,

$$\begin{aligned} \therefore \int_{1500}^{2500} \frac{A}{x^3} dx &= 1 \Rightarrow A \int_{1500}^{2500} x^{-3} dx = 1 \Rightarrow A \left[\frac{x^{-2}}{-2} \right]_{1500}^{2500} = 1 \\ &\Rightarrow -\frac{A}{2} \left[\frac{1}{(2500)^2} - \frac{1}{(1500)^2} \right] = 1 \Rightarrow -\frac{A}{20000} \left[\frac{1}{625} - \frac{1}{225} \right] = 1 \\ &\Rightarrow -\frac{A}{20000} \left[\frac{9-25}{5625} \right] = 1 \Rightarrow 16A = 5625 \times 20000 \\ &\Rightarrow A = \frac{5625 \times 20000}{16} = 5625 \times 1250 = 7031250. \end{aligned}$$

$$\begin{aligned} \text{Now, } P[1600 \leq X \leq 2000] &= \int_{1600}^{2000} f(x) dx = A \int_{1600}^{2000} \frac{1}{x^3} dx \\ &= -\frac{A}{2} \left[\frac{1}{x^2} \right]_{1600}^{2000} = -\frac{A}{2} \left[\frac{1}{(2000)^2} - \frac{1}{(1600)^2} \right] \\ &= -\frac{A}{20000} \left[\frac{1}{400} - \frac{1}{256} \right] = -\frac{A}{20000} \left[\frac{16-25}{6400} \right] \\ &= \frac{9 \times 7031250}{20000 \times 6400} = \frac{2025}{4096} \end{aligned}$$

5) i) As the given distribution is probability distribution,

$\therefore$ sum of all the probabilities = 1

$$\Rightarrow k + 3k + 5k + 7k + 9k + 11k + 13k + 15k + 17k = 1$$

$$\Rightarrow 81k = 1 \Rightarrow k = \frac{1}{81}$$

The distribution function of X is given in the following table:

X	F(x) = P[X ≤ x]
0	$k = \frac{1}{81}$
1	$k + 3k = 4k = \frac{4}{81}$
2	$4k + 5k = 9k = \frac{9}{81}$
3	$9k + 7k = 16k = \frac{16}{81}$
4	$16k + 9k = 25k = \frac{25}{81}$
5	$25k + 11k = 36k = \frac{36}{81}$
6	$36k + 13k = 49k = \frac{49}{81}$
7	$49k + 15k = 64k = \frac{64}{81}$
8	$64k + 17k = 81k = 1$

6) For $x < 0$,

$$F(x) = P[X \leq x] = \int_{-\infty}^x f(x) dx = \int_{-\infty}^x 0 dx = 0 \quad [\because f(x) = 0 \text{ for } x < 0].$$

For $0 \leq x < 1$,

$$\begin{aligned} F(x) = P[X \leq x] &= \int_{-\infty}^x f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^x f(x) dx \quad [\because 0 \leq x < 1] \\ &= 0 + \int_0^x \frac{x}{2} dx \quad \left[\because f(x) = \frac{x}{2} \text{ for } 0 \leq x < 1 \right] \\ &= \frac{1}{2} \left[\frac{x^2}{2} \right]_0^x = \frac{x^2}{4}. \end{aligned}$$

For $1 \leq x < 2$,

$$\begin{aligned} F(x) = P[X \leq x] &= \int_{-\infty}^x f(x) dx \\ &= \int_{-\infty}^0 f(x) dx + \int_0^1 f(x) dx + \int_1^x f(x) dx \\ &= 0 + \int_0^1 \frac{x}{2} dx + \int_1^x \frac{1}{2} dx = \frac{1}{4} [x^2]_0^1 + \frac{1}{2} [x]_1^x \end{aligned}$$

$$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2} = \frac{1}{4}(2x-1)$$

For $2 \leq x < 3$,

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(x) dx \\ &= \int_{-\infty}^0 f(x) dx + \int_0^1 f(x) dx + \int_1^2 f(x) dx + \int_2^x f(x) dx \\ &= 0 + \int_0^1 \frac{x}{2} dx + \int_1^2 \frac{1}{2} dx + \int_2^x \frac{1}{2}(3-x) dx \\ &= \left[\frac{x^2}{4} \right]_0^1 + \frac{1}{2} [x]_1^2 + \frac{1}{2} \left[3x - \frac{x^2}{2} \right]_2^x \\ &= \frac{1}{4} + \frac{1}{2} + \frac{1}{2} \left[\left(3x - \frac{x^2}{2} \right) - (6-2) \right] \\ &= \frac{1}{2} \left(3x - \frac{x^2}{2} \right) - \frac{5}{4} = -\frac{x^2}{4} + \frac{3x}{2} - \frac{5}{4} \end{aligned}$$

For $3 \leq x < \infty$,

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(x) dx \\ &= \int_{-\infty}^0 f(x) dx + \int_0^1 f(x) dx + \int_1^2 f(x) dx + \int_2^3 f(x) dx + \int_3^x f(x) dx \\ &= 0 + \int_0^1 \frac{x}{2} dx + \int_1^2 \frac{1}{2} dx + \int_2^3 \frac{1}{2}(3-x) dx + \int_3^x 0 dx \\ &= \left[\frac{x^2}{4} \right]_0^1 + \left[\frac{x}{2} \right]_1^2 + \frac{1}{2} \left[3x - \frac{x^2}{2} \right]_2^3 + 0 \\ &= \frac{1}{4} + \left(1 - \frac{1}{2} \right) + \frac{1}{2} \left[\left(9 - \frac{9}{2} \right) - (6-2) \right] \\ &= \frac{1}{4} + \frac{1}{2} + \frac{1}{2} \left(\frac{9}{2} - 4 \right) \\ &= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1 \end{aligned}$$

Hence, the distribution function is given by:

$$F(x) = \begin{cases} 0, & -\infty < x < 0 \\ \frac{x^2}{4}, & 0 \leq x < 1 \\ \frac{2x-1}{4}, & 1 \leq x < 2 \\ -\frac{x^2}{4} + \frac{3x}{2} - \frac{5}{4}, & 2 \leq x < 3 \\ 1, & 3 \leq x < \infty \end{cases}$$

Check Your Progress 2

- 1) Let us compute the marginal totals. Thus, the complete table with marginal totals is given as

X \ Y	1	2	3	p(x)
1	$\frac{1}{12}$	0	$\frac{1}{18}$	$\frac{1}{12} + 0 + \frac{1}{18} = \frac{5}{36}$
2	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{4}$	$\frac{1}{6} + \frac{1}{9} + \frac{1}{4} = \frac{19}{36}$
3	0	$\frac{1}{5}$	$\frac{2}{15}$	$0 + \frac{1}{5} + \frac{2}{15} = \frac{1}{3}$
p(y)	$\frac{1}{4}$	$\frac{14}{45}$	$\frac{79}{180}$	1

Therefore,

- i) Marginal distribution of X is

X	p(x)
1	$5/36$
2	$19/36$
3	$1/3$

- ii) $P[X + Y < 5]$

$$\begin{aligned}
 &= P[X = 1, Y = 1] + P[X = 1, Y = 2] + P[X = 1, Y = 3] \\
 &\quad + P[X = 2, Y = 1] + P[X = 2, Y = 2] + P[X = 3, Y = 1] \\
 &= \frac{1}{12} + 0 + \frac{1}{18} + \frac{1}{6} + \frac{1}{9} + 0 = \frac{15}{36}.
 \end{aligned}$$

2) First compute the marginal totals, then you will be able to find

$$\text{i) } P[X = 4] = \frac{1}{4}, \text{ and hence}$$

$$P[Y = 2 | X = 4] = \frac{P[Y = 2, X = 4]}{P[X = 4]} = \frac{2}{5}$$

$$\text{ii) } P[Y = 2] = \frac{2}{5}$$

3) The conditional density function of Y given X is $f(y|x) = \frac{f(x,y)}{f(x)}$,

where $f(x,y)$ is the joint density function, which is given; and $f(x)$ is the marginal density function which, by definition, is given by

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} f(x,y) dy \\ &= \int_0^1 f(x,y) dy \quad [\because 0 \leq y < 1] \\ &= \int_0^1 (x+y) dy \\ &= \left[xy + \frac{y^2}{2} \right]_0^1 \\ &= \left[x(1) + \frac{(1)^2}{2} - 0 \right] = x + \frac{1}{2}, \quad 0 \leq x < 1. \end{aligned}$$

$\therefore$ the conditional density function of Y given X is

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{x+y}{x+\frac{1}{2}}, \text{ for } 0 \leq x < 1 \text{ and } 0 \leq y < 1.$$

Check Your Progress 3

1) Let X be the amount (in rupees) won by you.

$\therefore$ X can take the values 100, 0 with $P[X = 100] = P[\text{Head}] = \frac{1}{2}$, and

$$P[X = 0] = P[\text{Tail}] = \frac{1}{2}.$$

$\therefore$ probability distribution of X is

X:	100	0
p(x)	$\frac{1}{2}$	$\frac{1}{2}$

and hence the expected amount won by you is

$$E(X) = 100 \times \frac{1}{2} + 0 \times \frac{1}{2} = 50.$$

2) Let X be the number of tosses till tail turns up.

$\therefore X$ can take values 1, 2, 3, 4... with

$$P[X = 1] = P[\text{Tail in the first toss}] = \frac{1}{2}$$

$$P[X = 2] = P[\text{Head in the first and tail in the second toss}] = \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^2,$$

$$P[X = 3] = P[\text{HHT}] = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^3, \text{ and so on.}$$

$\therefore$ Probability distribution of X is

$X:$	1	2	3	4	5...
$p(x)$	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^5 \dots$

and hence

$$E(X) = 1 \times \frac{1}{2} + 2 \times \left(\frac{1}{2}\right)^2 + 3 \times \left(\frac{1}{2}\right)^3 + 4 \times \left(\frac{1}{2}\right)^4 + \dots \quad \dots (1)$$

Multiplying both sides by $\frac{1}{2}$, we get

$$\begin{aligned} \frac{1}{2}E(X) &= \left(\frac{1}{2}\right)^2 + 2 \times \left(\frac{1}{2}\right)^3 + 3 \times \left(\frac{1}{2}\right)^4 + 4 \times \left(\frac{1}{2}\right)^5 + \dots \\ \Rightarrow \frac{1}{2}E(X) &= \left(\frac{1}{2}\right)^2 + 2 \times \left(\frac{1}{2}\right)^3 + 3 \times \left(\frac{1}{2}\right)^4 + \dots \quad \dots (2) \end{aligned}$$

[Shifting the position one step towards right so that we get the terms having same power at the same positions as that in (1)]

Now, subtracting (2) from (1), we have

$$\begin{aligned} E(X) - \frac{1}{2}E(X) &= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \dots \\ \Rightarrow \frac{1}{2}E(X) &= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \dots \\ \Rightarrow E(X) &= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots \end{aligned}$$

(Which is an infinite G.P. with first term $a = 1$ and
common ratio $r = \frac{1}{2}$)

$$= \frac{1}{1 - \frac{1}{2}} \quad [\because S_{\infty} = \frac{a}{1-r} \text{ (see Unit 3 of course MST - 001)}]$$

$$= \frac{1}{\frac{1}{2}} = 2.$$

$$\begin{aligned} 3) \quad E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{-\infty}^0 x f(x) dx + \int_0^1 x f(x) dx + \int_1^2 x f(x) dx + \int_2^{\infty} x f(x) dx \\ &= \int_{-\infty}^0 x(0) dx + \int_0^1 x(x^3) dx + \int_1^2 x(2-x)^3 dx + \int_2^{\infty} x(0) dx \\ &= 0 + \int_0^1 x^4 dx + \int_1^2 x[8 - x^3 - 6x(2-x)] dx + 0 \\ &= \int_0^1 x^4 dx + \int_1^2 (8x - x^4 - 12x^2 + 6x^3) dx \\ &= \left[\frac{x^5}{5} \right]_0^1 + \left[8\frac{x^2}{2} - \frac{x^5}{5} - 12\frac{x^3}{3} + 6\frac{x^4}{4} \right]_1^2 \\ &= \frac{1}{5} + \left[\left\{ \frac{8(2)^2}{2} - \frac{(2)^5}{5} - \frac{12(2)^3}{3} + \frac{6(2)^4}{4} \right\} - \left\{ \frac{8(1)^2}{2} - \frac{(1)^5}{5} - \frac{12(1)^3}{3} + \frac{6(1)^4}{4} \right\} \right] \\ &= \frac{1}{5} + \left[\left\{ 16 - \frac{32}{5} - 32 + 24 \right\} - \left\{ 4 - \frac{1}{5} - 4 + \frac{3}{2} \right\} \right] \\ &= \frac{1}{5} + \left[\frac{8}{5} - \frac{13}{10} \right] = \frac{1}{5} + \frac{3}{10} = \frac{1}{2}. \end{aligned}$$

4) As X is a random variable with mean μ ,

$$\therefore E(X) = \mu \quad \dots (1)$$

$$\text{Now, } E(Z) = E\left(\frac{X - \mu}{\sigma}\right)$$

$$= E\left[\frac{1}{\sigma}(X - \mu)\right]$$

$$= \frac{1}{\sigma} E[X - \mu] \quad [\text{Using Property 2 of Expectation Sec 5.7}]$$

$$\begin{aligned}
 &= \frac{1}{\sigma} [E(X) - \mu] \quad [\text{Using Property 3 of Expectation Sec 5.7}] \\
 &= \frac{1}{\sigma} [\mu - \mu] \quad [\text{Using Property 1 of Expectation Sec 5.7}] \\
 &= 0
 \end{aligned}$$

Note: Mean of standard random variable is zero.

Check Your Progress 4

1) Variance of standard random variable $Z = \frac{X - \mu}{\sigma}$ is given as

$$\begin{aligned}
 V(Z) &= V\left(\frac{X - \mu}{\sigma}\right) = V\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right) \\
 &= V\left[\frac{1}{\sigma}X + \left(-\frac{\mu}{\sigma}\right)\right] \\
 &= \left(\frac{1}{\sigma}\right)^2 V(X) \\
 &= \frac{1}{\sigma^2} V(X) \\
 &= \frac{1}{\sigma^2} (\sigma^2) = 1 \quad \left[\because \text{it is given that the standard deviation of } X \text{ is } \sigma \text{ and hence its variance is } \sigma^2 \right]
 \end{aligned}$$

Note: The mean of standard random variate is '0' and its variance is 1.

2) Given that $E(Y) = 0 \Rightarrow E(aX - b) = 0 \Rightarrow aE(X) - b = 0$

$$\Rightarrow a(10) - b = 0$$

$$\Rightarrow 10a - b = 0 \quad \dots (1)$$

Also as $V(Y) = 1$,

hence $V(aX - b) = 1$

$$\Rightarrow a^2 V(X) = 1 \Rightarrow a^2 (25) = 1 \Rightarrow a^2 = \frac{1}{25}$$

$$\Rightarrow a = \frac{1}{5} \quad [\because a \text{ is positive}]$$

$\therefore$ From (1), we have

$$10\left(\frac{1}{5}\right) - b = 0 \Rightarrow 2 - b = 0 \Rightarrow b = 2$$

Hence, $a = \frac{1}{5}$, $b = 2$.

5.11 REFERENCES

IGNOU Course Material

1. Post-Graduate Diploma In Applied Statistics (PGDAST), MST 003: Probability Theory, Block 2: Random Variables and Expectation, UNIT 5: Random Variables (for Sections 5.2 to 5.5 and related portions in Sections 5.0 , 5.1, 5.9, 5.10 and Check Your Progress questions)
2. PGDAST, MST 003: Probability Theory, Block 2: Random Variables and Expectation, UNIT 6: Bivariate Discrete Random Variables (for Section 5.6 and related portions in Sections 5.0 , 5.1, 5.9, 5.10 and Check Your Progress questions)
3. PGDAST, MST 003: Probability Theory, Block 2: Random Variables and Expectation, UNIT 7: Bivariate Continuous Random Variables (for Section 5.6 and related portions in Sections 5.0 , 5.1, 5.9, 5.10 and Check Your Progress questions)
4. PGDAST, MST 003: Probability Theory, Block 2: Random Variables and Expectation, UNIT 8: Mathematical Expectation (for Section 5.7, 5.8 and related portions in Sections 5.0 , 5.1, 5.9, 5.10 and Check Your Progress questions)

Reference Book

5. “Introduction to Probability and Statistics for Engineers and Scientists”, 3rd Edition, Sheldon M. Ross, Elsevier Academic Press, 2004